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Some properties of the (q, h)-binomial coefficients

Zhizheng Zhang†‡ and Jun Wang‡

† Department of Mathematics, Luoyang Teachers College, Luoyang, Henan 471022, People's Republic of China

‡ Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, People's Republic of China

E-mail: junwang@dlut.edu.cn

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Abstract. A further analogue of Newton's binomial formula was introduced in the (q, h)-deformed quantum plane by Benaoum, leading to a more generalized analysis, called a (q, h)-analysis. The purpose of this paper is to establish some further interesting properties for (q, h)-analysis. The (q, h)-analogue of the multinomial formula, the (q, h)-reciprocal formula and (q, h)-analogues of Chu's and Vandermonde's identities are obtained.

The q-analysis is an extension of the ordinary analysis by the addition of an extra parameter q. In [1], the h-analogue of Newton's binomial formula was introduced, leading to a new analysis, called a h-analysis.

Recently, in [2], a further analogue of Newton's binomial formula was introduced in the (q, h)-deformed quantum plane, leading to a more generalized analysis, called a (q, h)-analysis. With this generalization, the q-analysis, h-analysis and ordinary analysis are recovered by taking $h = 0, q = 1$ and $(q = 1, h = 0)$, respectively. Benaoum considered Manin's q-plane $x'y' = qy'x'$, by the following linear transformation (see [3] and references therein):

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \frac{h}{q-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Manin's q-plane changes to

$$xy = qyx + hy^2.$$

Even though the linear transformation is singular for $q = 1$, the resulting quantum plane is well defined. Benaoum obtained the (q, h)-Newton binomial formula:

$$(x + y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} y^k x^{n-k}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)}$ are called (q, h)-binomial coefficients, given as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} = \begin{bmatrix} n \\ k \end{bmatrix}_q h^k (h^{-1})_{[k]}$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_{(q,h)} = 1$ and $(a)_{[k]} = \prod_{j=0}^{k-1}(a+[j]_q)$, and $\begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)}$ obeys the following properties, $1 < k < n$:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{(q,h)} = q^k \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} + (1+h[k-1]_q) \begin{bmatrix} n \\ k-1 \end{bmatrix}_{(q,h)}$$

and

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{(q,h)} = (1+h[k]_q)\frac{[n+1]_q}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)}.$$

The purpose of this paper is to establish some further interesting properties for (q,h)-analysis.

First of all, we give several propositions, which are to be used later. These propositions can be proved by induction. Here we choose not to state their proofs.

Proposition 1.

$$x^m y^n = \sum_{l=0}^{m} \frac{[m]_q!}{[m-l]_q!} \sum_{r_1+\cdots+r_n=l} q^{mn-nr_1-(n-1)r_2-\cdots-r_n} h^l y^{n+l} x^{m-l}.$$

It is a generalization of the following result.

Corollary. *(Benaoum [2])*

$$x^k y = \sum_{r=0}^{k} \frac{[k]_q!}{[k-r]_q!} q^{k-r} h^r y^{r+1} x^{k-r}$$

and

$$xy^k = q^k y^k x + h[k]_q y^{k+1}.$$

Proposition 2.

$$y^k x = \frac{1}{q^k}(xy^k - h[k]_q y^{k+1})$$

and

$$yx^k = \sum_{i=0}^{k} C(k,i)[i]_q! h^i x^{k+1-i} y^{i+1}$$

where $C(k,i)$ are defined by

$$C(k,0) = \frac{1}{q^k}$$

$$C(k,1) = -[k]_q \frac{1}{q^{2k-1}}$$

and

$$C(k+1,i) = \frac{1}{q^{i+1}}(C(k,i) - qC(k,i-1)).$$

Proposition 3.

$$y^m x^n = \sum_{l=0}^{n} \sum_{r_1+\cdots+r_m=l} C(n,r_1)C(n-r_1,r_2)\ldots C(n-r_1-r_2-\cdots-r_{m-1},r_m)$$
$$\times [r_1]_q![r_2]_q!\ldots[r_m]_q! h^l x^{m-l} y^{m+l}.$$

By successive use of proposition 3 it is easy to obtain another form of the (q, h)-Newton binomial formula:

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} \sum_{l=0}^{n-k} \sum_{r_1+\cdots+r_m=l} C(n-k, r_1)C(n-k-r_1, r_2) \times \ldots C(n-k-r_1-r_2-\cdots-r_{k-1}, r_k)[r_1]_q![r_2]_q!\ldots[r_k]_q!h^l x^{n-k-l} y^{k+l}.$$

Using proposition 1 and induction, there is an immediate generalization to m variables.

Theorem 1. *(The (q, h)-analogue of the multinomial formula)*

$$(x_1+x_2+\cdots+x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix}_{(q,h)} x_m^{k_1} x_{m-1}^{k_2}, \ldots, x_1^{k_m}$$

where $x_i x_j = q x_j x_i + h x_j^2$ if and only if $i < j$, and the (q, h)-multinomial coefficients are given as

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix}_{(q,h)} = \frac{[n]_q!}{[k_1]_q![k_2]_q!, \ldots, [k_m]_q!} \prod_{i=1}^{m-1} \prod_{j=1}^{k_i-1} (1+(m-i)h[j]_q)$$

where $k_1+k_2+\cdots+k_m = n$. The (q, h)-multinomial formula in turn implies the recurrence relation for the (q, h)-multinomial coefficients:

$$\begin{aligned} \begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix}_{(q,h)} &= \begin{bmatrix} n-1 \\ k_1-1, k_2, \ldots, k_m \end{bmatrix}_{(q,h)} (1+(m-1)h[k_1-1]_q) \\ &+ \begin{bmatrix} n-1 \\ k_1, k_2-1, \ldots, k_m \end{bmatrix}_{(q,h)} q^{k_1}(1+(m-2)h[k_2-1]_q) \\ &+ \begin{bmatrix} n-1 \\ k_1, k_2, k_3-1, \ldots, k_m \end{bmatrix}_{(q,h)} q^{k_1+k_2}(1+(m-3)h[k_3-1]_q) + \cdots \\ &+ \begin{bmatrix} n-1 \\ k_1, k_2, k_3, \ldots, k_{m-1}-1, k_m \end{bmatrix}_{(q,h)} q^{k_1+k_2+\cdots+k_{m-2}}(1+h[k_{m-1}-1]_q) \\ &+ \begin{bmatrix} n-1 \\ k_1, k_2, k_3, \ldots, k_{m-1}, k_m-1 \end{bmatrix}_{(q,h)} q^{k_1+k_2+\cdots+k_{m-1}}. \end{aligned}$$

Corollary. *(Johnson [5])*

$$(x_1+x_2+\cdots+x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix}_{(q)} x_m^{k_1} x_{m-1}^{k_2}, \ldots, x_1^{k_m}$$

where $x_i x_j = q x_j x_i$ if and only if $i < j$, and the Gaussian multinomial coefficients are given as follows:

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix}_{(q)} = \frac{[n]_q!}{[k_1]_q![k_2]_q!, \ldots, [k_m]_q!}$$

where $k_1+k_2+\cdots+k_m = n$.

Theorem 2.

$$\sum_{v=0}^{n} \begin{bmatrix} n \\ v \end{bmatrix}_{(q,h)} G_{(q,h)}(v, k) = \sum_{v=0}^{n} G_{(q,h)}(n, v) \begin{bmatrix} v \\ k \end{bmatrix}_{(q,h)} = \delta_{n,k}$$

where $G_{(q,h)}(n, k)$ are defined by

$G_{(q,h)}(n,k) = 0$ *if* $n < k$, $G_{(q,h)}(n,0) = 1$,

$$G_{(q,h)}(n+1,k) = \frac{1}{1+h[n]_q} G_{(q,h)}(n,k-1) - \frac{q^n}{1+h[n]_q} G_{(q,h)}(n,k).$$

Proof.

$$\begin{aligned}
G_{(q,h)}(n,k-1) &= \sum_{i=0}^{n} G_{(q,h)}(n,i)\delta_{i+1,k} \\
&= \sum_{i=0}^{n} G_{(q,h)}(n,i) \sum_{j=0}^{i+1} \begin{bmatrix} i+1 \\ j \end{bmatrix}_{(q,h)} G_{(q,h)}(j,k) \\
&= \sum_{i=0}^{n} G_{(q,h)}(n,i) \sum_{j=0}^{i+1} \left(q^j \begin{bmatrix} i \\ j \end{bmatrix}_{(q,h)} + (1+h[j-1]_q) \begin{bmatrix} i \\ j-1 \end{bmatrix}_{(q,h)} \right) G_{(q,h)}(j,k) \\
&= \sum_{j=0}^{n} (q^j G_{(q,h)}(j,k) + (1+h[j]_q) G_{(q,h)}(j+1,k)) \sum_{i=j}^{n} G_{(q,h)}(n,i) \begin{bmatrix} i \\ j \end{bmatrix}_{(q,h)} \\
&= \sum_{j=0}^{n} (q^j G_{(q,h)}(j,k) + (1+h[j]_q) G_{(q,h)}(j+1,k)) \delta_{nj} \\
&= q^n G_{(q,h)}(n,k) + (1+h[n]_q) G_{(q,h)}(n+1,k).
\end{aligned}$$

□

Theorem 3. *(The (q,h)-binomial reciprocal formula) Let $\{a_k\}$ and $\{b_k\}$ be any two sequences of complex numbers. Then the following reciprocal formula holds:*

$$a_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} b_k$$

$$b_n \prod_{j=0}^{n} (1+[j-1]_q h) = \sum_{k=0}^{n} (-1)^{n-k} \frac{q^{\frac{1}{2}(n-k)(n-k-1)}}{\prod_{j=0}^{k}(1+[j-1]_q h)} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} a_k.$$

Proof. It is easily seen that the sequences

$$(-1)^{n-k} \frac{q^{\frac{1}{2}(n-k)(n-k-1)}}{\prod_{i=0}^{n}(1+[i-1]_q h)\prod_{j=0}^{k}(1+[j-1]_q h)} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)}$$

satisfy the recurrence relation of $G_{(q,h)}(n,k)$ in theorem 2. □

Corollary 3.1. *The binomial reciprocal formula, Gaussian binomial reciprocal formula and h-binomial reciprocal formula are the special cases of the (q,h)-binomial reciprocal formula.*

Theorem 4. *(The (q,h)-analogue of symmetrical and trinomial modified identities of binomial coefficients, see [4])*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} = \frac{(1+[1]_q h)(1+[2]_q h)\ldots(1+[k-1]_q h)}{(1+[1]_q h)(1+[2]_q h)\ldots(1+[n-k]_q h)} \begin{bmatrix} n \\ n-k \end{bmatrix}_{(q,h)}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} \begin{bmatrix} k \\ l \end{bmatrix}_{(q,h)} = \frac{(1+[1]_q h)(1+[2]_q h)\ldots(1+[k-1]_q h)}{(1+[1]_q h)(1+[2]_q h)\ldots(1+[k-l]_q h)} \begin{bmatrix} n \\ l \end{bmatrix}_{(q,h)} \begin{bmatrix} n-l \\ k-l \end{bmatrix}_{(q,h)}.$$

Proof. Apply $\begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} = \begin{bmatrix} n \\ k \end{bmatrix}_q h^k (h^{-1})_{[k]}$. □

Theorem 5. *(The (q, h)-analogue of Chu's identity)*

$$\sum_{i=0}^{n-k} q^{ik}(1 + h[k-1]_q) \begin{bmatrix} n-1-i \\ k-1 \end{bmatrix}_{(q,h)} = \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)}.$$

Proof. Apply the recurrence relation of (q, h)-binomial coefficients. □

Theorem 6.

$$\sum_{i=0}^{k} q^{i(n-k)} \frac{(1 + [1]_q h)(1 + [2]_q h) \dots (1 + [k]_q h)}{(1 + [1]_q h)(1 + [2]_q h) \dots (1 + [k-i]_q h)} \begin{bmatrix} n-1-i \\ k-i \end{bmatrix}_{(q,h)} = \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)}.$$

Proof. Apply theorems 4 and 5. □

Theorem 7. *(The (q, h)-analogue of Vandermonde's identity)*

$$\begin{bmatrix} n+m \\ i \end{bmatrix}_{(q,h)} = \sum_{k+r+l=i} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} \begin{bmatrix} m \\ r \end{bmatrix}_{(q,h)} \frac{[n-k]_q!}{[n-k-l]_q!} \sum_{t_1+\dots+t_r=l} q^{(n-k)r - rt_1 - (r-1)t_2 - \dots - t_r} h^l.$$

Proof.

$$\begin{aligned}
\sum_{i=0}^{n+m} \begin{bmatrix} n+m \\ i \end{bmatrix}_{(q,h)} y^i x^{n+m-i} &= (x+y)^{n+m} \\
&= (x+y)^n (x+y)^m \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} y^k x^{n-k} \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_{(q,h)} y^r x^{m-r} \\
&= \sum_{k=0}^{n} \sum_{r=0}^{m} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} \begin{bmatrix} m \\ r \end{bmatrix}_{(q,h)} y^k x^{n-r} y^r x^{m-r} \\
&= \sum_{k=0}^{n} \sum_{r=0}^{m} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} \begin{bmatrix} m \\ r \end{bmatrix}_{(q,h)} y^k \\
&\quad \times \sum_{l=0}^{n-k} \frac{[n-k]_q!}{[n-k-l]_q!} \sum_{t_1+\dots+t_n=l} q^{(n-k)r - rt_1 - (r-1)t_2 - \dots - t_r} h^l y^{r+l} x^{n-k-l} x^{m-r} \\
&= \sum_{k=0}^{n} \sum_{r=0}^{m} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} \begin{bmatrix} m \\ r \end{bmatrix}_{(q,h)} \\
&\quad \times \sum_{l=0}^{n-k} \frac{[n-k]_q!}{[n-k-l]_q!} \sum_{t_1+\dots+t_n=l} q^{(n-k)r - rt_1 - (r-1)t_2 - \dots - t_r} h^l y^{k+r+l} x^{n+m-k-r-l} \\
&= \sum_{i=0}^{n+m} \Bigg(\sum_{k+r+l=i} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} \begin{bmatrix} m \\ r \end{bmatrix}_{(q,h)} \\
&\quad \times \frac{[n-k]_q!}{[n-k-l]_q!} \sum_{t_1+\dots+t_r=l} q^{(n-k)r - rt_1 - (r-1)t_2 - \dots - t_r} h^l \Bigg) y^i x^{n+m-i}.
\end{aligned}$$

□

Corollary 7.1.

$$\sum_{k+r+l=i} \begin{bmatrix} n \\ k \end{bmatrix}_{(q,h)} \begin{bmatrix} n \\ r \end{bmatrix}_{(q,h)} \frac{[n-k]_q!}{[n-k-l]_q!} \sum_{t_1+\cdots+t_r=l} q^{(n-k)r-rt_1-(r-1)t_2-\cdots-t_r} h^l = \begin{bmatrix} 2n \\ i \end{bmatrix}_{(q,h)}.$$

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